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Complete hypersurfaces with infinite fundamental group*

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1. Hypersurfaces with constant scalar curvature

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. In this section, we shall study curvature structures of complete hypersurfaces with constant scalar curvature in a unit sphere. First of all, we present several examples.

Example 1. *For any $0 < c < 1$, by considering the standard immersions*

$$S^{n-1}(c) \subset \mathbf{R}^n, \quad S^1(\sqrt{1-c^2}) \subset \mathbf{R}^2$$

and taking the Riemannian product immersion

$$S^1(\sqrt{1-c^2}) \times S^{n-1}(c) \hookrightarrow \mathbf{R}^2 \times \mathbf{R}^n,$$

we obtain a compact hypersurface $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, where $r = \frac{n-2}{nc^2} > 1 - \frac{2}{n}$.

We know that this hypersurface $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ has the following characterizations:

1. *$r > 1 - \frac{2}{n}$,*
2. *the number of its distinct principal curvatures is two.*
3. *its fundamental group is infinity.*

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Example 2. *By make using of the same construction as in example 1, we can obtain a compact hypersurface $S^k(c_1) \times S^{n-k}(c_2)$, $1 < k < n-1$, in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$. This hypersurface has finite fundamental group and the number of its distinct principal curvatures is two.*

Example 3. *We consider an isoparametric hypersurface M^6 in $S^7(1)$ with principal curvatures $\lambda_1 = \lambda_2 = \theta$, $\lambda_3 = \frac{\theta+1}{1-\theta}$, $\lambda_4 = \lambda_5 = -\frac{1}{\theta}$, $\lambda_6 = -\frac{1-\theta}{1+\theta}$, where $\theta = \sqrt{\frac{13+\sqrt{165}}{2}}$. This hypersurface M^6 satisfies $r = 1$ and the number of its distinct principal curvatures is four.*

In 1977, S.Y. Cheng and Yau [4] characterized compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. They proved

Theorem 1. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$. If $r \geq 1$ and the sectional curvatures of M are non-negative, then M is isometric to the totally umbilical hypersurface $S^n(c)$ or the Riemannian product $S^k(c_1) \times S^{n-k}(c_2)$ $1 \leq k \leq n-1$, where $S^k(c)$ denote the sphere of radius c.*

Proof. For a C^2-function f on M, we consider a differential operator $\Box$ defined by

$$\Box f = \sum_{i,j=1}^{n} (nH\delta_{ij} - h_{ij})\nabla_i \nabla_j f, \tag{1.1}$$

where h_{ij} and H are components of the second fundamental form and the mean curvature of M, respectively. Thus, we have

$$\Box nH = \sum_{i,j,k=1}^{n} h_{ijk}^2 - n^2\|\text{grad}H\| + \sum_{i,j=1}^{n} (\lambda_i - \lambda_j)^2 K_{ij}, \tag{1.2}$$

where λ_i's are principal curvatures and h_{ijk}'s denote components of the covariant differentiation of the second fundamental form. From $r \geq 1$, we can prove

$$\sum_{i,j,k=1}^{n} h_{ijk}^2 \geq n^2|\text{grad}H|^2. \tag{1.3}$$

Since M has non-negative sectional curvature, we have $K_{ij} \geq 0$. Hence, we infer

$$\Box nH \geq 0. \tag{1.4}$$

According to Stokes theorem, we know that H is constant and the number of distinct principal curvatures is at most two. Therefore, M is an isoparametric hypersurface with at most two distinct principal curvatures. From a theorem of Cartan, we know that theorem 1 is true. $\Box$

Further, by making use of the similar method which was used by Nakagawa and the author in [3] and the differential operator (1.1) introduced by S.Y. Cheng and Yau, Li [5] proved

Theorem 2. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$. If $r \geq 1$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to either the totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c^2 = \frac{n-2}{nr} \leq \frac{n-2}{n}$, where S is the squared norm of the second fundamental form of M.*

Proof. Since $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds, we can prove

$$\sum_{i,j=1}^{n} (\lambda_i - \lambda_j)^2 K_{ij} \geq 0.$$

From $r \geq 1$, we know that (1.3) is satisfies. Thus, we infer that the inequality (1.4) is true. Hence, theorem 2 is true by using the same assertion as in theorem 1. □

Remark 1. *In proofs of theorems 1 and 2, the estimate $\sum_{i,j,k=1}^{n} h_{ijk}^2 \geq n^2|gradH|^2$ is necessary. In order to prove it, the condition of $r \geq 1$ and the assumption of constant scalar curvature is essential. Hence, the condition $r \geq 1$ and the assumption of constant scalar curvature play an essential role in theorems 1 and 2.*

Remark 2. *From example 1, we know that some of $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ does not appear in these results of theorems 1 and 2 becuase some of them does not satisfy the condition $r \geq 1$.*

Moreover, Cheng [2] researched the inversed problem of example 1. The following was proved.

Theorem 3. *Let M be an n-dimensional complete hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If M has only two distinct principal curvatures one of which is simple, then, $r > 1 - \frac{2}{n}$ holds and M is isometric to $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ if $r \neq \frac{n-2}{n-1}$ and $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, where $c^2 = \frac{n-2}{nr}$.*

From the assertions above, it is natural and interesting to study the following:

Problem 1. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r > 1 - \frac{2}{n}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then is M isometric to the totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$?*

From theorem 2, we know that if $r \geq 1$, then the problem 1 was solved affirmatively. In [2], the author gave an affirmative answer for this problem when $r = \frac{n-2}{n-1}$. But for the other case, this problem seems to be a very hard problem.

Problem 2. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r > 1 - \frac{2}{n}$ and the sectional curvature is non-negative, then is M isometric to the totally umbilical hypersurface or the Riemannian product $S^k(\sqrt{c_1}) \times S^{n-k}(c_2)$, $1 \leq k \leq n-1$?*

2. Compact hypersurfaces with infinite fundamental group

In this section, we shall try to solve problems 1 and 2 introduced in the section 1. From example 1, we know that $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ has infinite fundamental group. We shall consider these problems under a topological condition. The following theorems will be proved.

Theorem 4. *Let M be an n-dimensional compact hypersurface with infinite fundamental group in $S^{n+1}(1)$. If $r \geq \frac{n-2}{n-1}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$, where $n(n-1)r$ is the scalar curvature of M and $c^2 = \frac{n-2}{nr}$.*

Proof. Since $r \geq \frac{n-2}{n-1}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, we infer

$$n + 2nH^2 - S \geq \frac{n-2}{\sqrt{n(n-1)}}\sqrt{n^2H^2(S-nH^2)}. \tag{2.1}$$

For any point p and any unit vector $\vec{u} \in T_pM$, we choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $e_n = \vec{u}$. From Gauss equation, we have

$$\operatorname{Ric}(\vec{u}) = (n-1) + nHh_{nn} - \sum_{i=1}^{n} h_{in}^2 \tag{2.2}$$

and we can prove

$$\operatorname{Ric}(\vec{u}) \geq \frac{n-1}{n}\{n + 2nH^2 - S - \frac{n-2}{\sqrt{n(n-1)}}\sqrt{n^2H^2(S-nH^2)}\}. \tag{2.3}$$

From (2.1), we have $\operatorname{Ric}(\vec{u}) \geq 0$. In particular, we can show that if $S < (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds, then $\operatorname{Ric}(\vec{u}) > 0$. Thus, if there exists a point p in M such that $S < (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then, at the point p, the Ricci curvature is positive. From the following Lemma 1 due to Aubin [1], we know that there exists a metric on M such that the Ricci curvature is positive on M. According to Myers theorem, we know that the fundamental group is finite. This is impossible because M has infinite fundamental group.

Lemma 1. (cf. Aubin [1, p. 344]). *If the Ricci curvature of a compact Riemannian manifold is non-negative and positive at somewhere, then the manifold carries a metric with positive Ricci curvature.*

Thus, we must have $S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$. And at each point, there exists a unit vector $\vec{u}$ such that $\operatorname{Ric}(\vec{u}) = 0$. Thus, we can conclude that M has only two distinct principal curvatures one of which is simple. Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field such that $h_{ij} = \lambda_i\delta_{ij}$, where λ_i's are principal curvatures on M. Without loss of generality, we can assume $\mu = \lambda_n$, $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$. From Gauss equation (2.2) and the definition of the Ricci curvature, we have $1 + \mu\lambda = 0$

because of $1 + \lambda_i \lambda_j = 1 + \lambda^2 > 0$, for any $i, j = 1, \cdots, n-1$. From Gauss equation, we have

$$\mu = \frac{n(r-1)}{2\lambda} - \frac{n-2}{2}\lambda.$$

Hence $\lambda^2 = \frac{n(r-1)+2}{n-2}$ and $\mu^2 = \frac{n-2}{n(r-1)+2}$.

We consider the integral submanifold of the corresponding distribution of the space of principal vectors corresponding to the principal curvature λ. Since the multiplicity of the principal curvature λ is greater than 1, we know that the principal curvature λ is constant on this integral submanifold (cf. Otsuki [6]). From $\lambda^2 = \frac{n(r-1)+2}{n-2}$ and $\mu^2 = \frac{n-2}{n(r-1)+2}$, we know that the scalar curvature $n(n-1)r$ and the principal curvature μ are constant. Thus, we obtain that M is isoparametric. Therefore, M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ because $S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds. This completes the proof of Theorem 4. □

Theorem 5. *Let M be an n-dimensional compact hypersurface with infinite fundamental group in $S^{n+1}(1)$. If the sectional curvatures are non-negative, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$.*

Proof. Since the sectional curvatures are non-negative, we have that the Ricci curvature is non-negative. From the arguments in the proof of theorem 4, we infer that at each point, there exists a unit vector $\vec{u}$ such that $\mathrm{Ric}(\vec{u}) = 0$.

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field such that $h_{ij} = \lambda_i \delta_{ij}$, where λ_i's are principal curvatures on M. Then, from Gauss equation, we have $1 + \lambda_i \lambda_j \geq 0$ for $i \neq j$. Further, there exists an i such that $\sum_{j \neq i}(1 + \lambda_i \lambda_j) = 0$ from the definition of Ricci curvature. Hence, we must have $1 + \lambda_i \lambda_j = 0$ for $j \neq i$. Therefore, M has only two distinct principal curvatures one of which is simple. Let $\mu = \lambda_i$ and $\lambda = \lambda_j$ for $j \neq i$. From Gauss equation, we have

$$\mu = \frac{n(r-1)}{2\lambda} - \frac{n-2}{2}\lambda. \tag{2.4}$$

Since $1 + \mu\lambda = 0$ and (2.4) hold, we have $\lambda^2 = \frac{n(r-1)+2}{n-2}$ and $\mu^2 = \frac{n-2}{n(r-1)+2}$. Hence, we have

$$S = (n-1)\lambda^2 + \mu^2 = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}.$$

By making use of the same assertion as in the proof of theorem 4, we infer that M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$. This completes the proof of Theorem 5. □

Remark 3. *In our theorems 4 and 5, we do not assume that the scalar curvature is constant. And in our theorem 5, we do not assume any condition on scalar curvature.*

References

[1] Aubin, T., Some nonlinear problems in Riemannian geometry, Springer-Verlag, Berlin, New York. 1998

[2] Cheng, Q.-M., Hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature, J. London Math. Soc., 64(2001), 755-768

[3] Cheng, Q.-M. and Nakagawa, H., Totally umbilical hypersurfaces, Hiroshima Math. J., 20(1990), 1-10

[4] Cheng, S. Y. and Yau, S. T., Hypersurfaces with constant scalar curvature, Math. Ann., 225(1977), 195-204.

[5] Li. H., Hypersurfaces with constant scalar curvature in space forms, Math. Ann., 305(1996), 665-672

[6] Otsuki, T., Minimal hypersurfaces in a Riemannian manifold of constant curvature, Amer. J. Math., 92(1970), 145-173